A Simple, Refined Theory for Bending and Stretching of Homogeneous Plates

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A theory of plates is developed that retains the simplicity of Reissner's theory, yet provides reliable three-dimensional information for stresses and displacements. It accounts for all major influences in a rational manner. Reissner's equations are obtained from the new equations for isotropic materials by averaging displacement variables over the plate thickness. Application of the theory to a benchmark problem demonstrates its capability to predict stresses with precision even for very thick plates.

Introduction

THE classical engineering theory of plate bending had its origin in the pioneering work of Sophie Germain. She was awarded a prize in 1815 for her attempt to provide a theoretical basis for the modal figures obtained in Chladni's vibration experiments.[1] Her work was finally published in 1821.[2] It contained an error in the strain energy of bending, which was corrected by Lagrange.[3] While the correct governing differential equation for flexural vibration of plates was independently established by Navier,[4] Poisson,[5] and Cauchy,[6] it was Kirchhoff[7] who resolved the famous controversy concerning the nature and number of proper boundary conditions. The theory reached maturity with the addition of Kirchhoff's boundary conditions.

This classical plate bending theory permits the satisfaction of two boundary conditions per edge. Three conditions, however, provide a more realistic behavioral description. The first theory with this property was developed by Reissner[8] in 1945. Reissner's theory accounts for transverse shear deformation and transverse normal stress effects. It was clarified later[9] and subsequently improved for bending without transverse normal stress.[10]

Theories developed specifically for dynamic applications that account for rotatory inertia and transverse shear deformation were later derived by Uflyand[11] and Mindlin.[12] These two theories differ only in the manner in which the shear deformation contributions are estimated. Transverse normal stress effects are ignored by these authors. A host of similar theories are surveyed by Panc.[13]

A significant advance was published by Reissner[10] in 1975. In this paper, a means of predicting true three-dimensional effects is presented that is based upon the use of the earlier Reissner equations.[8,9] Although restricted to bending without lateral loads (and hence transverse normal stress), excellent agreement with an exact elasticity solution is shown for a stress concentration problem.

Lo, et al.[14,15] presented a theory of isotropic and laminated composite plates specifically for problems that involve rapidly fluctuating loads over a charateristic length of the order of the plate thickness. Their displacement formulation is based upon a suitable assumption for a kinematically admissible displacement field. Stress equilibrium equations are not fully satisfied, and some of the stresses are poorly predicted. This led the authors to consider an alternative means for obtaining stress estimates.[16]

Recently, Cheng[17] provided a plate theory based upon the three-dimensional equations of elasticity. He did not consider any transverse loading in the development.

Krenk[18] constructed a hierarchy of plate theories in a systematic manner utilizing expansions of orthogonal polynomials. This work is very similar in concept to that of Lo, et al.[14,15] Krenk treats general issues and relates his findings to those of Cheng.[17] Since no specific problems are considered, the practical usefulness of his various equation sets remains undetermined.

The recent work of Rehfield and Murthy[19] on planar bending shows that two effects in addition to transverse shear strain arise naturally in considering improvements to classical theory. These additional influences are due to transverse normal strain and a nonclassical contribution to the axial stress. Earlier, Goodier[20] suspected that these influences, in addition to transverse shear, were important, but offered no means of estimating them quantitatively and no concrete examples of their influence. The matter has been clarified by the results presented in Ref. 19. In the present work, all of these effects are rationally accounted for.

This work has two primary objectives. The first is to develop new equations for static plate bending and stretching that are simple, consistent, and complete—all major influences are included in them. The fundamental ideas of Refs. 19 and 21 are the cornerstone of the development. Second, the equations are validated by demonstrating that Reissner's equations[8,9] can be obtained from them and by study of a classic benchmark problem for which an exact solution is available.

Summary of Classical Theory

First, classical theory will be reviewed to clearly identify the point of departure of the present work and to establish notation. The classical theory of plate bending is based upon the kinematic assumption of Kirchhoff and the assumption of approximate plane stress. The fundamental equations are summarized below. The notation adopted herein and the sign convention are shown in Fig. 1. To facilitate the presentation, the following subscript convention is used: 1) Latin indices will assume the values 1, 2 and 6, unless specifically noted otherwise; 2) Greek indices will assume the values 1 and 2; and 3) repeated indices in the same expression imply a sum of terms over the appropriate range of the index.

The stress components are identified in the conventional manner—$\sigma_{11}, \sigma_{22}, \sigma_{zz}, \sigma_{2z}, \sigma_{1z}$, and σ_{12}. An alternative engineering notation is to write the stresses, in the order given,

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as six entries in a vector:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \vdots \\ \sigma_{12} \end{Bmatrix} \tag{1}$$

A corresponding strain vector $\{\epsilon\}$ will be used also. In accordance with classical theory, the surface parallel stresses are determined by

$$\sigma_i = N_i/h + M_i z/\bar{I} \qquad (i=1,2,6) \tag{2}$$

where N_i and M_i are force and moment resultants, respectively.

$$N_i = \int_{-c}^{c} \sigma_i \mathrm{d}z \qquad (i=1,2,6) \tag{3}$$

$$M_i = \int_{-c}^{c} \sigma_i z \mathrm{d}z \qquad (i=1,2,6) \tag{4}$$

c is the plate semithickness, $h=2c$ is the thickness, and $\bar{I}$ is the moment of inertia per unit width, $h^3/12$.

Estimates for the transverse shear and normal stresses are determined from equilibrium considerations. The results are

$$\sigma_{\alpha z} = (Q_\alpha/2\bar{I})(c^2 - z^2) \tag{5}$$

$$\sigma_{zz} = (q/6\bar{I})(2c^3 + 3c^2 z - z^3) \tag{6}$$

In the above, Q_α are the transverse shear stress resultants defined as

$$Q_\alpha = \int_{-c}^{c} \sigma_{z\alpha} \mathrm{d}z \tag{7}$$

These stresses satisfy equilibrium and the following boundary conditions on the upper and lower surfaces:

$$\sigma_{\alpha z}(x_1,x_2,c) = \sigma_{\alpha z}(x_1,x_2,-c) = 0$$

$$\sigma_{zz}(x_1,x_2,c) = q, \sigma_{zz}(x_1,x_2,-c) = 0 \tag{8}$$

The force and moment resultants satisfy overall equilibrium equations:

$$N_{\gamma\alpha,\gamma} = 0 \tag{9}$$

$$M_{\gamma\alpha,\gamma} - Q_\alpha = 0 \tag{10}$$

$$Q_{\gamma,\gamma} + q = 0 \tag{11}$$

The resultants with two indices are the same as those defined in Eqs. (3) and (4) with the alternative designation for stresses adopted.

The displacement field is constrained by the Kirchhoff hypothesis and is linear through the plate thickness:

$$u_\alpha = U_\alpha(x_1,x_2) - zW_{,\alpha}(x_1,x_2) \tag{12}$$

$$w = W(x_1,x_2) \tag{13}$$

The Kirchhoff hypothesis is equivalent to requiring that

$$\gamma_{\alpha z} = u_{\alpha,z} + w_{,\alpha} = 0 \tag{14}$$

and

$$\epsilon_{zz} = w_{,z} = 0 \tag{15}$$

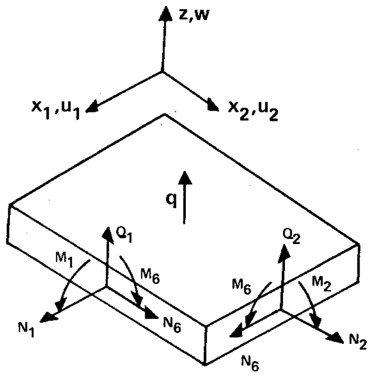

Fig. 1 Notation and sign convention.

For a general anisotropic, Hookean elastic plate, the stresses and strains are related by Generalized Hooke's Law.

$$\sigma_i = C_{ij}\gamma_j \qquad (i,j=1,2,3,\ldots,6) \tag{16}$$

$$\epsilon_i = S_{ik}\sigma_k \qquad (i,k=1,2,3,\ldots,6) \tag{17}$$

The C_{ij} terms are stiffnesses and the S_{ij} terms are flexibilities. In classical theory, the plane stress versions of Eqs. (16) and (17) are utilized.

Foundations of a New Theory

Overview

Classical theory is based upon kinematic assumptions that are equivalent to ignoring, and hence setting to zero, transverse normal strain and the two transverse shear strain components. These conditions are specified in Eqs. (14) and (15). The central assumption that replaces this hypothesis in the present development is that the statically equivalent stresses obtained from classical theory can be used to estimate the transverse normal strain and transverse shear strains. These strain estimates permit a suitable displacement field to be determined, which further leads to improved stresses. The development parallels that of Ref. 19.

Kinematics

The foregoing assumption is adopted with the classical stress estimates provided by Eqs. (2), (5), and (6). Also, attention is restricted to specially orthotropic materials for which Eq. (17) simplifies to the following:

$$w_{,z} = \epsilon_{zz} = S_{3\alpha}\sigma_\alpha + \underline{S_{33}\sigma_{zz}} \tag{18}$$

$$u_{1,z} + w_{,1} = \gamma_{1z} = S_{55}\sigma_{1z} \tag{19}$$

$$u_{2,z} + w_{,2} = \gamma_{2z} = S_{44}\sigma_{2z} \tag{20}$$

The underlined term in Eq. (18) will be neglected in the subsequent development. In the usual plate bending situations, the transverse normal stress is two orders of magnitude smaller than the surface parallel stresses. This term adds a negligible contribution under these conditions, which are the conditions of concern here.

With the aid of the above simplification, Eq. (18) is integrated to yield

$$w = W(x_1,x_2) + S_{3\alpha}\left(\frac{N_\alpha}{h}z + \frac{M_\alpha}{\bar{I}}\frac{z^2}{2}\right) \tag{21}$$

where W is the lateral deflection of the plate midsurface. The displacement components u_1 and u_2 are found in an analogous manner using Eqs. (19) and (20). The results are

$$u_\alpha = U_\alpha(x_1,x_2) - zW_{,\alpha} + \frac{\bar{Q}_\alpha}{2\bar{I}}\left(c^2 z - \frac{z^3}{3}\right)$$

$$- S_{3\beta}\left(\frac{N_{\beta,\alpha}}{h}\frac{z^2}{2} + \frac{M_{\beta,\alpha}}{\bar{I}}\frac{z^3}{6}\right) \tag{22}$$

The notation

$$\bar{Q}_1 = S_{55}Q_1, \quad \bar{Q}_2 = S_{44}Q_2 \tag{23}$$

has been utilized in the above.

Equations (21) and (22) comprise the desired displacement distribution. These equations should be compared with Eqs. (12) and (13) for classical Kirchhoff theory. Equation (22) includes section warping effects—a parabolic nonclassical stretching contribution and a cubic term that corresponds to bending-like behavior. These effects are determined explicitly by the present approach and arise from estimates of precisely the strains that are ignored in classical theory. The midsurface displacement components, U_1, U_2, and W, emerge as natural kinematic variables.

Refined Surface Parallel Stresses

The surface parallel stresses σ_1, σ_2, and σ_6 are generally the largest and most important stress components. An accurate knowledge of them is often all that is needed in a practical application. Refined estimates that improve Eq. (2) are central to the theoretical improvements that are sought.

As a preliminary step in obaining improved surface parallel stresses, Eq. (17) for the surface parallel strains is specialized for a specially orthotropic material and inverted. The result is

$$\sigma_i = \bar{C}_{ij}\epsilon_j + \bar{C}_i\sigma_{zz} \tag{24}$$

The coefficients are defined to be

$$\bar{C}_{11} = S_{22}/S, \quad \bar{C}_{12} = \bar{C}_{21} = -S_{12}/S, \quad \bar{C}_{22} = S_{11}/S$$

$$\bar{C}_{66} = 1/S_{66}, \quad \bar{C}_{16} = \bar{C}_{61} = \bar{C}_{26} = \bar{C}_{62} = 0$$

$$\bar{C}_1 = (S_{12}S_{23} - S_{13}S_{22})/S, \quad \bar{C}_2 = (S_{21}S_{13} - S_{23}S_{11})/S,$$

$$\bar{C}_6 = 0 \tag{25}$$

where

$$S = S_{11}S_{22} - (S_{12})^2 \tag{26}$$

The surface parallel strains can be estimated using the displacement components given in Eq (22). The process is facilitated by introducing the differential operator matrix $L_{i\alpha}$.

$$L_{11} = (\ )_{,1}, \qquad L_{12} = 0$$

$$L_{21} = 0, \qquad L_{22} = (\ )_{,2}$$

$$L_{61} = (\ )_{,2}, \qquad L_{62} = (\ )_{,1} \tag{27}$$

The strains can now be simply expressed as

$$\epsilon_i = L_{i\alpha}u_\alpha \tag{28}$$

With the aid of Eq. (22), the above becomes,

$$\epsilon_i = \epsilon_i^{(0)} + z\epsilon_i^{(1)} + \frac{z^2}{2}\epsilon_i^{(2)} + \frac{z^3}{6}\epsilon_i^{(3)} \tag{29}$$

The strain parameters are defined as

$$\epsilon_i^{(0)} = L_{i\alpha}U_\alpha \tag{30}$$

$$\epsilon_i^{(1)} = L_{i\alpha}\left(\frac{c^2}{2\bar{I}}\bar{Q}_\alpha - W_{,\alpha}\right) \tag{31}$$

$$\epsilon_i^{(2)} = -L_{i\alpha}(S_{3\beta}N_{\beta,\alpha}/h) \tag{32}$$

$$\epsilon_i^{(3)} = -L_{i\alpha}[(\bar{Q}_\alpha + S_{3\beta}M_{\beta,\alpha})/\bar{I}] \tag{33}$$

where $\epsilon_i^{(0)}$ are extensional strains at the plate midsurface, and $z\epsilon_i^{(1)}$ are bending and twisting strains. These strains have counterparts in classical theory. The remaining parameters are new. The functions $\epsilon_i^{(2)}$ will be called nonclassical stretching strains. Analogously, $\epsilon_i^{(3)}$ are nonclassical bending strains.

The strains in Eq. (29), together with Eqs. (24) and (6), provide improved estimates for the surface parallel stresses. These stresses, however, must be used in the overall equivalence requirements (3) and (4) for force and moment resultants.

The force and moment resultants are

$$N_i = h\bar{C}_{ij}\left(\epsilon_j^{(0)} + \frac{c^2}{6}\epsilon_j^{(2)}\right) + \bar{C}_i q/c \tag{3a}$$

$$M_i = \bar{I}\bar{C}_{ij}\left(\epsilon_j^{(1)} + \frac{c^2}{10}\epsilon_j^{(3)}\right) + \bar{C}_i\frac{2c^2}{5}q \tag{4a}$$

These equations allow the surface parallel stresses to be expressed in terms of force and moment resultants:

$$\sigma_i = \frac{N_i}{h} + \frac{M_i}{\bar{I}}z + \underline{\frac{1}{2h}\bar{C}_{ij}\epsilon_j^{(2)}\left(z^2 - \frac{c^2}{3}\right)}$$

$$+ \underline{\frac{1}{6\bar{I}}(\bar{C}_{ij}\epsilon_j^{(3)} - \bar{C}_i q)\left(z^3 - \frac{3c^2}{5}z\right)} \tag{34}$$

The underlined terms are the nonclassical surface parallel stress contributions, which are the desired refinements. The first group listed are associated with stretching, the second with bending.

Summary

The governing equations for the new theory can now be summarized. They encompass four main categories. Overall plate-type equations consist of the equilibrium equations (9-11) and the constitutive equations (3a) and (4a). In the latter, the strain parameters are defined in Eqs. (30-33). In addition, two sets of equations provide the distributions of stresses and displacements throughout the structure. The set for stresses consists of Eqs. (5), (6), and (34). Displacements are given in Eqs. (21) and (22).

The above collection of equations requires the specification of boundary conditions. It will be shown subsequently that by suitably averaging the response parameters through the thickness, the equations of Reissner's plate theory[8-10] for isotropic materials can be obtained from the stress and displacement distributions of the present theory. It appears that, on the basis of this relationship, one may safely apply boundary conditions analogous to those of Reissner's theory.

There are five boundary conditions per edge, two associated with stretching and three with bending and twisting of the plate midsurface. While the modeling of boundary restraint conditions in elementary theories is straightforward, since the displacement varies linearly through the thickness, explicit section warping does not permit, for example, a unique, fully satisfactory definition of a fixed edge. More experience with the use of the equations and specific study of the sensitivity of predictions to boundary restraint modeling are required. In the interim, the work of Reissner[8-10] and the general discussion of boundary effects by Goldenveiser[22] serve as guides.

The development of the present equations is distinctive in that the classical Kirchhoff displacement assumption is abandoned and no variational principle is utilized. This permits a flexibility that results in a simple means of including nonclassical effects in an explicitly determined way. The order of the equations is not raised in the process. This is a distinct advantage. A disadvantage, however, is the fact that consistently derived boundary conditions are not automatically obtained as in a variational formulation.

The authors have obtained a set of variationally derived equations based upon the stresses defined by Eqs. (5), (6), and (34). The nonclassical surface parallel stress effects do not appear in the resulting constitutive equations and the order of the equations is higher. The simpler equations presented herein are believed to offer practical advantages in applications.

Relation to Reissner's Plate Theory

Introductory Remarks

This section has a single purpose—to show that the fundamental equations of Reissner's plate theory for isotropic materials can be obtained from the present equations specialized in the same manner. This is a significant step in validation as the consistency and correctness of Reissner's theory is well established.

Reissner[8] develops equations for the bending of isotropic plates using a complementary energy principle. The equations include the influence of transverse shear and normal stresses. By virtue of the approach utilized, weighted or averaged kinematic variables are defined[9] and utilized. It will be shown that the Reissner equations can be obtained from the present theory by constructing the Reissner variables using the displacement expressions (21) and (22).

Establishment of the Relationship

Reissner's transverse displacement variable $\tilde{W}$ is defined as

$$\tilde{W}=\frac{1}{2\bar{I}}\int_{-c}^{c} w(c^2-z^2)\,\mathrm{d}z \tag{35a}$$

With the aid of Eq. (21), the above can be expressed in terms of the present variables for an isotropic material as

$$\tilde{W}=W-\frac{3}{10}\frac{\nu}{Eh}(M_1+M_2) \tag{35b}$$

where ν is Poisson's ratio and E is Young's modulus.

Two rotation variables, $\tilde{\phi}_1$ and $\tilde{\phi}_2$, are also utilized by Reissner. They are determined from

$$\tilde{\phi}_\alpha=\frac{1}{\bar{I}}\int_{-c}^{c} u_\alpha z\,\mathrm{d}z \tag{36a}$$

Equation (22) permits the above to be written for an isotropic material as

$$\tilde{\phi}_\alpha=-W_{,\alpha}+\frac{3\nu}{10Eh}(M_1+M_2)_{,\alpha}+\frac{6}{5}\frac{Q_\alpha}{Gh} \tag{36b}$$

$G=E/2(1+\nu)$ is the shear modulus.

Equations (35b) and (36b) permit one of Reissner's fundamental relationships to be derived in a straightforward manner. If Eq. (35b) is differentiated with respect to x_α and added to Eq. (36b), the following result emerges:

$$\tilde{W}_{,\alpha}+\tilde{\phi}_\alpha=\frac{6}{5}\frac{Q_\alpha}{Gh} \tag{37}$$

This is the relation between transverse shear stress resultants and weighted transverse shear strain variables ($\tilde{W}_{,\alpha}+\tilde{\phi}_\alpha$) in Reissner's plate theory.[8-10]

For an isotropic material

$$u_{1,1}=(1/E)(\sigma_1-\nu\sigma_2-\nu\sigma_{zz}) \tag{38}$$

If this equation is multiplied by z, integrated over the plate thickness, and divided by $\bar{I}$, the following result is obtained:

$$\tilde{\phi}_{1,1}=\frac{1}{E\bar{I}}\left(M_1-\nu M_2-\frac{2}{5}c^2\nu q\right) \tag{39}$$

In an analogous way, an expression involving the other rotation variable is constructed.

$$\tilde{\phi}_{2,2}=\frac{1}{E\bar{I}}\left(M_2-\nu M_1-\frac{2}{5}c^2\nu q\right) \tag{40}$$

Equations (39) and (40) may be inverted to express the moment resultants in terms of the rotation variables.

$$M_1=D(\tilde{\phi}_{1,1}+\nu\tilde{\phi}_{2,2})+\frac{2c^2}{5}\frac{\nu}{(1-\nu)}q \tag{41}$$

$$M_2=D(\tilde{\phi}_{2,2}+\nu\tilde{\phi}_{1,1})+\frac{2c^2}{5}\frac{\nu}{(1-\nu)}q \tag{42}$$

where $D=E\bar{I}/(1-\nu^2)$ is the plate bending stiffness per unit width. M_6 is identical in both theories.

Summary

Equations (37), (41), and (42) are the fundamental relations of Reissner's plate theory. These equations are supplemented by the overall equilibrium equations, Eqs. (9-11), and boundary conditions. It has been demonstrated, therefore, that Reissner's equations can be obtained from the present theory by appropriately introducing averaged kinematic variables and by averaging certain equations.

It is noted that while Reissner's theory is obtainable from the present one, the reverse process does not logically follow. This is because a knowledge of Reissner's variables does not permit the determination of the response throughout the structure. However, since the present theory is now available, Eqs. (35b) and (36b) permit Reissner's variables to be transformed such that the three-dimensional nature of the response can be discerned by equations of the present theory. In effect, any solution to Reissner's equations, provided that the boundary conditions are meaningful in the transformed context, is a solution to the present equations.

Simply Supported Plate under Sinusoidal Loading

Introductory Remarks

A quantitative demonstration for the new equations is provided by the solution to a classical benchmark problem. The problem is the response of a rectangular plate to sinusoidally distributed loading. An exact solution for this case is given by Pagano.[23] This is a generic problem which is often used as a test case.

Predictions of the present theory are compared with the exact solution[23] and predictions utilizing Reissner's equations,[8,9] Lo, Christensen and Wu's theory[16,17] (LCW) and classical plate theory (CPT). An approach for presenting results is illustrated that yields the range of validity of the theory as a function of plate geometry.

Analysis and Results

Consider a rectangular, specially orthotropic plate of length $2a$, width $2b$, and thickness $2c$. Let the origin of coordinates be the geometric center of the plate. The applied loading is defined by the following boundary conditions at the upper and lower plate surfaces:

$$\sigma_{1z}(x_1,x_2,C)=0,\quad \sigma_{1z}(x_1,x_2,-C)=0 \tag{43}$$

$$\sigma_{2z}(x_1,x_2,C)=0, \quad \sigma_{2z}(x_1,x_2,-C)=0 \tag{44}$$

$$\sigma_{zz}(x_1,x_2,C)=q_0\cos\frac{\pi x_1}{2a}\cos\frac{\pi x_2}{2b}$$

$$\sigma_{zz}(x_1,x_2,-C)=0 \tag{45}$$

The elastic properties, expressed in terms of engineering constants, are taken to be those chosen by Pagano,[23] which are representative of modern graphite-epoxy composite materials.

$$E_{11}=172.37 \text{ GPa } (25\times10^6 \text{ psi})$$

$$E_{22}=E_{33}=6.89 \text{ GPa } (1\times10^6 \text{ psi})$$

$$G_{13}=G_{12}=3.45 \text{ GPa } (0.5\times10^6 \text{ psi})$$

$$G_{23}=1.38 \text{ GPa } (0.2\times10^6 \text{ psi})$$

$$\nu_{12}=\nu_{13}=0.25, \quad \nu_{23}=0.25 \tag{46}$$

where the E are Young's moduli, the G are shear moduli, and the ν are Poisson's ratios. These properties are utilized for the purpose of comparison with earlier numerical results. In reality, ν_{23} should be different from ν_{12} and ν_{13}.

Solutions to the above problem have been obtained using the present theory and the other theories cited previously. Although the loading distribution in the form of one-half sinusoidal wavelength in each direction is considered explicitly, the results are applicable to the case of general loading of the form $\cos(m\pi x_1/2a)\cos(n\pi x_2/2b)$, where m and n are integers. This follows from the fact that each half wavelength may be considered separately by an appropriate reduction in plate length or width. At the plate edges, the following boundary conditions are satisfied:

$$\text{At } x_1=\pm a: \quad \sigma_{11}=u_2=w=0 \tag{47}$$

$$\text{At } x_2=\pm b: \quad \sigma_{22}=u_1=w=0 \tag{48}$$

The plate response is sinusoidal in nature. The variables have the general form

$$W=W_0\cos(\pi x_1/2a)\cos(\pi x_2/2b)$$

$$M_\alpha=M_{0\alpha}\cos(\pi x_1/2a)\cos(\pi x_2/2b)$$

$$M_6=M_{06}\sin(\pi x_1/2a)\sin(\pi x_2/2b)$$

$$\vdots \tag{49}$$

Algebraic relationships among the amplitude parameters W_0, M_0, M_{06},... are deterimined from the governing equations.

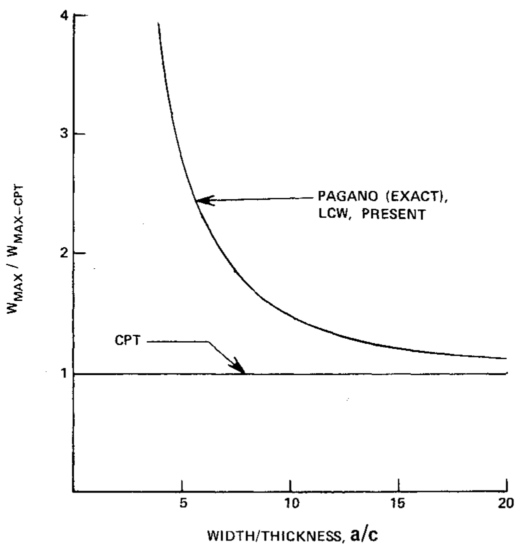

Fig. 2 Variation of the maximum transverse deflection parameter.

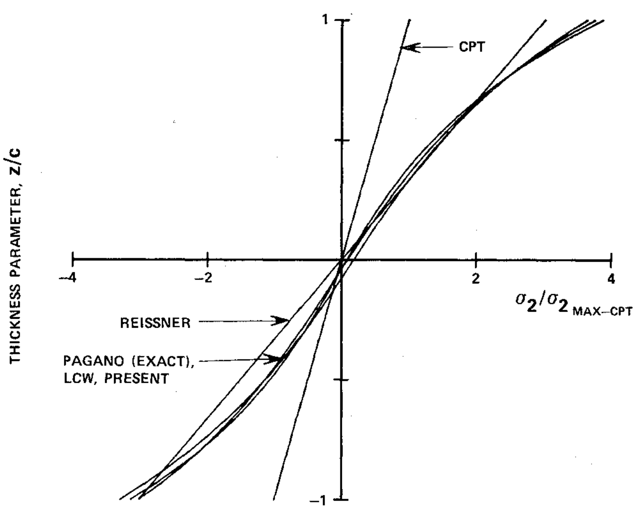

Fig. 4 Distribution of axial stress in the x_2 direction through the plate thickness, width/thickness = 3.

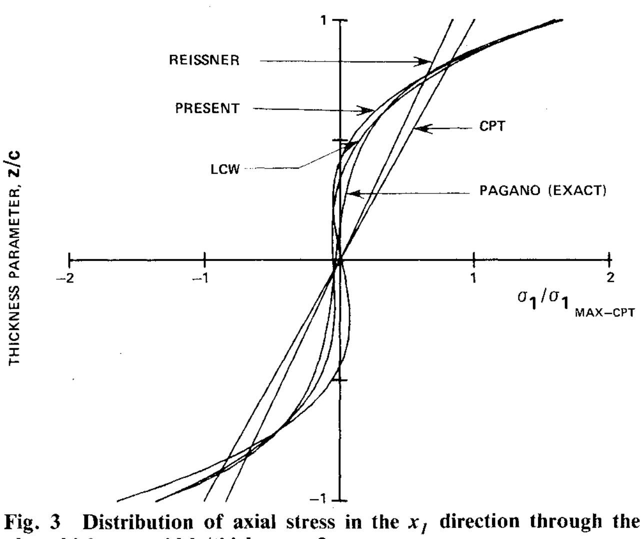

Fig. 3 Distribution of axial stress in the x_1 direction through the plate thickness, width/thickness = 3.

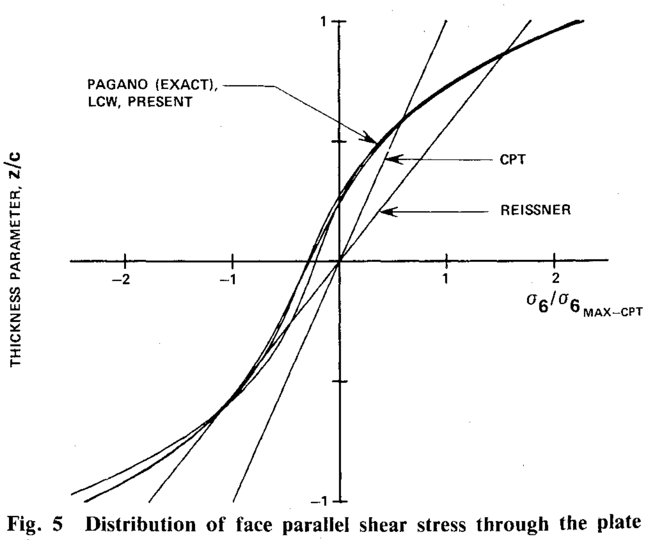

Fig. 5 Distribution of face parallel shear stress through the plate thickness, width/thickness = 3.

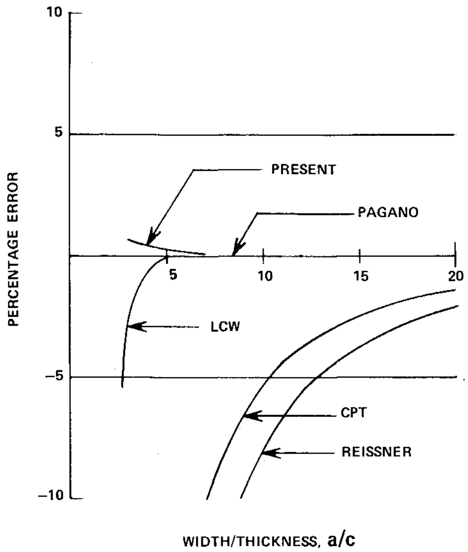

Fig. 6 Error analysis of maximum axial stress, x_1 direction.

Representative results for square plates, $a=b$, appear in Figs. 2-6. A maximum midsurface transverse deflection ratio is presented in Fig. 2. W_{max} is the maximum midsurface transverse deflection. The subscript "CPT" denotes a result from classical plate theory. Similarly, "LCW" refers to Lo, Christensen, and Wu's theory. This figure illustrates the fact that nonclassical influences are important for plates with the assumed elastic properties and that all the theories are capable of predicting the deflection accurately.

A very stringent test of predictive capability is provided by the distribution of face parallel stresses for a plate with $a/c=3.0$. The axial stress in the x_1 direction, axial stress in the x_2 direction, and the shear stress appear in Figs. 3-5. These figures clearly illustrate the precise detail achievable with the present theory. The LCW theory predictions are also excellent. These equations are much more difficult to apply than the present ones in most applications, however, as they involve 11 equations and 11 unknown variables.

The relative merits of each theory under consideration are assessed on the basis of the percentage error in the maximum value of σ_1 with respect to the exact solution in Fig. 6. For the present purposes, a 5% error is assumed to be an acceptable limit. The range of plate width-to-thickness ratios in which the error is less than 5% is considered the range of validity for the theory. The point at which a theoretical prediction just exceeds the limit is a cutoff or limit value of the width-to-thickness ratio.

Only the σ_1 study is presented here, however, any quantity of interest can be studied on the basis of the approach just described. Figure 6 provides evidence of the ability of the present theory to predict stresses in ranges of width-to-thickness previously thought to require three-dimensional numerical treatment.

Concluding Remarks

A new theory for plates has been developed and applied. It is characterized by the simplicity of Reissner-type theory, yet it accounts for all major nonclassical influences—transverse shear strains, transverse normal strain, and nonclassical face parallel stresses with their concomitant section warping—in a rational manner. It is demonstrated that Reissner's equations can be obtained from the new equations for isotropic materials by averaging displacement variables over the thickness. Application of the theory to a benchmark problem has demonstrated its capability to predict stresses with precision even for very thick plates.

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